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1980 J. Phys. A: Math. Gen. 13 L23

(http://iopscience.iop.org/0305-4470/13/2/004)

J. Phys. A: Math. Gen., **13** (1980) L23–L25. Printed in Great Britain

LETTER TO THE EDITOR

Electrovac solution

K C Das

Department of Physics, Katwa College, Katwa, Burdwan, India

Received 30 October 1979

Abstract. A new static electrovac solution of the Einstein–Maxwell equation is derived from the physically realistic, asymptotically flat rotating vacuum solution of Kinnersley and Chitre.

Kinnersley and Chitre (1978a) presented a two-parameter metric which they claimed to be a new, physically realistic, asymptotically flat, rotating vacuum metric different from the $\delta = 2$ metric of Tomimatsu and Sato (1973). This also reduces to the Weyl $\delta = 2$ static solution when the twist potential is switched off by making a certain constant zero. In an earlier paper (Das and Banerji 1978) we showed explicitly the formal similarity between stationary gravitational fields and static electrovac fields in prolate spheroidal coordinates and put forward a scheme for obtaining the latter from the former. In this Letter we have utilised the scheme, slightly modified, and obtained the static electrovac solution of the Einstein–Maxwell equations from a new stationary gravitational field given by Kinnersley and Chitre (1978a). The electrovac solution is functionally non-related (a rare class) and asymptotically flat referring to a massive electric dipole.

Starting with the axially symmetric stationary line element

$$ds^2 = -e^{-u}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] + e^u(dt - \omega\, d\phi)^2 \tag{1}$$

we can write the field equations in Ernst's notation (1968a),

$$\mathrm{Re}(\mathscr{G})\nabla^2\mathscr{G} = \nabla\mathscr{G}\,.\,\nabla\mathscr{G} \tag{2}$$

where $\mathscr{G} = e^u + i\phi$ is the 'Ernst potential' and

$$\omega_1 = \rho\, e^{-2u}\phi_2 \qquad \omega_2 = -\rho\, e^{-2u}\phi_1. \tag{3}$$

The solution of equation (2) determines the metric (1) uniquely (Tomimatsu and Sato 1973). Equation (2) expressed in prolate spheroidal coordinates defined by

$$\rho = (x^2-1)^{1/2}(1-y^2)^{1/2} \qquad z = xy \tag{4}$$

takes the form

$$(x^2-1)u_{11} + (1-y^2)u_{22} + 2xu_1 - 2yu_2 = -e^{-2u}[(x^2-1)\phi_1^2 + (1-y^2)\phi_2^2] \tag{5}$$

$$(x^2-1)\phi_{11} + (1-y^2)\phi_{22} + 2x\phi_1 - 2y\phi_2 = 2[(x^2-1)u_1\phi_1 + (1-y^2)u_2\phi_2] \tag{6}$$

where suffixes 1 and 2 denote partial derivatives with respect to x and y, respectively.

The axially symmetric static electrovac metric

$$ds^2 = e^{2\alpha}\, dt^2 - e^{-2\alpha}[e^{2\gamma'}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] \tag{7}$$

0305-4470/80/020023+03\$01.00

may be determined uniquely if we can solve the following equations. Here also we use prolate spheroidal coordinates and notation as given by Ernst (1968b):

$$(x^2-1)\alpha_{11}+(1-y^2)\alpha_{22}+2x\alpha_1-2y\alpha_2=e^{-2\alpha}[(x^2-1)\psi_1^2+(1-y^2)\psi_2^2] \tag{8}$$

$$(x^2-1)\psi_{11}+(1-y^2)\psi_{22}+2x\psi_1-2y\psi_2=2[(x^2-1)\alpha_1\psi_1+(1-y^2)\alpha_2\psi_2] \tag{9}$$

where

$$\mathcal{G}=e^{2\alpha}-\mathcal{E}\mathcal{E}^* \qquad \mathcal{E}=A_4+iA_3' \qquad \psi=A_4 \text{ or } A_3'. \tag{10}$$

A_4 is the scalar electrostatic potential defined by Ernst (1968b) and A_3' is connected with the scalar magnetic potential A_3 by the relations

$$A_{3,1}=-e^{-2\alpha}(1-y^2)A'_{3,2} \qquad A_{3,2}=e^{-2\alpha}(x^2-1)A'_{3,1}. \tag{11}$$

An inspection of equations (5), (6) and (8), (9) reveals a formal similarity except for a difference of sign in equations (5) and (8). A change of parameter technique enables us to obtain a solution of equations (8) and (9) from the known solution of equations (5) and (6).

Kinnersley and Chitre (1978a) have given the solution of equations (5) and (6) in their own notation, and when it is transformed to the familiar notation of Ernst (1968a), we get the above solution as follows:

$$e^u=1-4A/B \qquad \phi=-4\beta yC/B \tag{12}$$

where β is a constant in Kinnersley and Chitre's solution and A, B and C are the following lengthy expressions:

$$A=x(x^2-1)[(x+1)^2(x^2-1)-\beta^2(x^2-y^2)^2]-2\beta^2y^2(x^2-y^2)(x+1)(x^2-2x+y^2)$$
$$B=[(x+1)^2(x^2-1)-\beta^2(x^2-y^2)^2]^2+4\beta^2y^2(x+1)^2(x^2-2x+y^2)^2$$
$$C=(x^2-y^2)[(x+1)^2(x^2-1)-\beta^2(x^2-y^2)^2]+2x(x^2-1)(x+1)(x^2-2x+y^2). \tag{13}$$

With the help of the parameter change technique (Das and Banerji 1978) we write down the lengthy solution of the static electrovac metric (7) which satisfies equations (8) and (9). To verify this by direct substitution of those lengthy expressions no doubt takes a lot of time, and at the same time is very tedious and laborious.

$$e^\alpha=1-4D/E \qquad \psi=4\beta yF/E \tag{14}$$

where

$$\begin{aligned}D=&(x^7+2x^6-x^5-4x^4-x^3+2x^2+x)+\beta^2(x^7-x^5)-\beta^2y^2(2x^4+2x^3)\\&-\beta^2(-x^3y^4-3xy^4+2xy^6+2y^6-4x^2y^4)\\E=&(x^8+4x^7+4x^6-4x^5-10x^4-4x^3+4x^2+4x+1)\\&+2\beta^2(x^8+2x^7-2x^5-x^4)+2\beta^2y^2(-3x^4y^2+2x^3y^2+6xy^2-y^2\\&-4x^6-4x^3-6x^2-2y^4+6x^4+12x^2y^2-4xy^4-2x^2y^4)\\&+\beta^4(x^8-4x^6y^2+6x^4y^4-4x^2y^6+y^8)\\F=&x^2(3x^4-6x^2+3)-y^2(-x^4+2x^2-1)+\beta^2(x^6-y^6-3x^4y^2+3x^2y^4).\end{aligned} \tag{15}$$

The above electrovac solution (15) is well behaved in the sense that $e^{2\alpha}\to 1$ and $\psi\to 0$ when $r\to\infty$, r being a new coordinate defined by

$$ax=r-m/2 \qquad y=\cos\theta. \tag{16}$$

The equilibrium shape of the pulsar, a rotating astronomical object where the general relativistic effect is not negligible, is an oblate spheroid. These are dense rotating stars with large magnetic fields. Hence, axially symmetric solutions of the Einstein–Maxwell field equations are of much interest in astrophysics. A lot of functionally dependent solutions are available in the literature but only a few functionally independent solutions are known. The above solution (15) is new, functionally independent and well behaved at spatial infinity.

With the transformation (16), $e^{2\alpha}$ and ψ take the following forms when expanded asymptotically:

$$
\begin{aligned}
e^{2\alpha} &= 1 - \frac{8a}{1+\beta^2}\frac{1}{r} + \frac{4a[8a - m(1+\beta^2)]}{(1+\beta^2)^3}\frac{1}{r^2} + O(r^{-3}) + \ldots \\
\psi &= \frac{4\beta(3+\beta^2)a^2}{(1+\beta^2)^2}\frac{\cos\theta}{r^2} + O(r^{-3}) + \ldots
\end{aligned}
\tag{17}
$$

Thus the above solution is asymptotically flat and represents a source of mass $4a/(1+\beta^2)$ and a dipole moment $4\beta(3+\beta^2)a^2/(1+\beta^2)^2$. When $\beta = 0$, the electrostatic potential vanishes and we get Weyl's static vacuum solution of the Einstein equation for $\delta = 2$ as in the case of an electromagnetic analogue of the TS, $\delta = 2$ solution (Das and Banerji 1978). Directional singularity at the poles $x = 1$, $y = \pm 1$ is already present in the Weyl static metric and the directional properties of the metric (15) are not much different from the TS metric studied by Economon and Ernst (1976) and Alcano (1976).

Kinnersley and Chitre (1978b) have also given a more general five-parameter solution of the stationary gravitational fields than the $\delta = 2$ solution of Tomimatsu and Sato and we may obtain accordingly the electrovac solution from this solution by the parameter change technique given. As the general solution is more lengthy than the $\delta = 2$ solution of Tomimatsu and Sato or the solution given in equation (12) the procedure is expected to be more difficult.

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